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On the Determination of the Properties of the Nodal Curve of a Unicursal Ruled Surface.

BY CHARLES H. SISAM.

The parametric equations of a unicursal ruled surface may be written:

$$x_i = a_i(\mu) + vb_i(\mu), \qquad i = 1, 2, 3, 4. \tag{1}$$

wherein $a_i(\mu)$, $b_i(\mu)$ are polynomials in μ.

I shall suppose—it being no restriction on the surface—that there exists no value of μ which satisfies all the equations:

$$a_i(\mu)b_j(\mu) - a_j(\mu)b_i(\mu) = 0, \qquad i, j = 1, 2, 3, 4.$$

The generator along which μ has any particular value μ_1 will be referred to as the generator μ.

Any point of the nodal curve is determined by two pairs of values (μ, v) and (μ', v') of the parameters. Hence, if from the four equations,

$$a_i(\mu) + vb_i(\mu) = \rho\,[a_i(\mu') + v'b_i(\mu')] \qquad i = 1, 2, 3, 4, \tag{2}$$

we eliminate ρ, μ', v', solve for v (or μ) and substitute in (1) we obtain the parametric equations of the nodal curve.

If, on the other hand, we eliminate v, ρ, v' from (2), we obtain:

$$\begin{vmatrix} a_1(\mu) & a_1(\mu') & b_1(\mu) & b_1(\mu') \\ a_2(\mu) & a_2(\mu') & b_2(\mu) & b_2(\mu') \\ a_3(\mu) & a_3(\mu') & b_3(\mu) & b_3(\mu') \\ a_4(\mu) & a_4(\mu') & b_4(\mu) & b_4(\mu') \end{vmatrix} = 0. \tag{3}$$

This expression is divisible by $(\mu - \mu')^2$. The resulting equation is expressible in terms of $\mu + \mu'$ and $\mu\mu'$ only. Call it $F(\mu + \mu', \mu\mu') = 0$. Put $\mu + \mu' = \xi$ and $\mu\mu' = \eta$ and consider $F(\xi, \eta) = 0$ as the non-homogeneous equation of a curve whose current co-ordinates are (ξ, η). It then follows that:

Each irreducible component of $F(\xi, \eta) = 0$ *determines an irreducible component of the nodal curve of the surface. If the latter is double only, then the number* *of times it is cut by an arbitrary generator equals the order, and its genus equals the genus, of the component of $F=0$ which determines it.*

For, let $\Phi(\xi, \mu)=0$ be an irreducible component of $F=0$. Let $p_{ik}(\mu) \equiv a_i(\mu)b_k(\mu)-b_i(\mu)a_k(\mu)$ be the line coordinates of the generator μ and $p'_{ik}(\mu')$ be those of the generator μ'. Then when $\Phi(\mu+\mu', \mu\mu')=0$, the curve:

$$(4) \quad \begin{cases} x_1 = (p_{12}p'_{13}-p_{13}p'_{12})(p_{34}p'_{42}-p_{42}p'_{34}) = \rho f_1(\mu+\mu', \mu\mu') \\ x_2 = (p_{42}p'_{23}-p_{23}p'_{42})(p_{34}p'_{14}-p_{14}p'_{34}) = \rho f_2(\mu+\mu', \mu\mu') \\ x_3 = (p_{34}p'_{13}-p_{13}p'_{34})(p_{34}p'_{42}-p_{42}p'_{34}) = \rho f_3(\mu+\mu', \mu\mu') \\ x_4 = (p_{34}p'_{14}-p_{14}p'_{34})(p_{34}p'_{42}-p_{42}p'_{34}) = \rho f_4(\mu+\mu', \mu\mu') \end{cases}$$

belongs to the double curve of the surface (1).

When the curve (4) is a double curve, only, on the surface, a one to one correspondence exists between its points and those of $\Phi(\xi, \eta)=0$, for any two generators intersecting on (4) determine a single point (their intersection) on (4), and also a single point (the point $\xi=\mu+\mu'$, $\eta=\mu\mu'$) on $\Phi=0$. Hence $\Phi=0$ and the double curve (4) are of the same genus. Moreover, if in $\Phi(\mu+\mu', \mu\mu')=0$ we give μ an arbitrary fixed value, we obtain m (= order of $\Phi(\xi, \eta)=0$) values for μ', hence the generator μ meets m other generators on (4) and therefore meets (4) in m points.

Two distinct components of $F(\xi, \eta)=0$ can not determine the same *double* curve on the surface (1). $F=0$ may, however, have one, or more, other components identical with a given component $\Phi=0$. The two sheets of the surface through the corresponding double curve (4) then touch, osculate, etc., according to the number of components of $F=0$, identical with $\Phi=0$. If Φ^i is a factor of F then the corresponding curve (4) counts i times as a component of the nodal curve.

A simple illustration (already obtained by another method by Snyder*) is that of a quintic scroll with an oscnodal conic. Writing the equations of the scroll in the form:

$$x_1=1; \quad x_2=\mu^2+v; \quad x_3=\mu^3-v\mu; \quad x_4=v\mu^2,$$

the equation $F(\xi, \eta)=0$ becomes $\xi^3=0$. The oscnodal conic is $4x_1x_4=x_2^2$, $x_3=0$. It counts for a nodal curve of order six.

A component of the nodal curve which is of multiplicity greater than two on the surface may, on the other hand, be determined by more than one component of the curve $F = 0$. Between the points of an i-fold curve on the surface and the totality of distinct components of $F = 0$ which determine it exists a $\left[1, \frac{i(i-1)}{2}\right]$ correspondence. These curves may be tacnodal, oscnodal, etc., under the same conditions as before. They exhibit, also, the following further singularity: Among the components of $F = 0$ which determine the same curve on the surface, some may enter to the first power only, others to the second, etc.; correspondingly, among the sheets of the surface passing through the curve some will simply intersect, others touch, etc. The equation $F = 0$, therefore, forms, as before, a criterion for the mode of intersection of the sheets of the surface along the curve.

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*Bulletin Am. Math. Soc., vol. 11, p. 182.

Consider, for example, a scroll having a fourfold conic along which the sheets of the surface touch in pairs. Such a surface may be obtained by setting up a (1, 2) correspondence between the points of the conic and the planes through the tangent at the point, then joining the point of tangency to the points in which the corresponding planes cut another conic. To the former conic corresponds in $F = 0$, a quadratic factor and a linear factor squared. It counts for sixteen as a component of the nodal curve.

A more interesting illustration is that of a scroll having a fourfold quartic, along which two sheets of the surface touch, the remaining intersections being simple. Such a surface is the nonic:

$$\begin{aligned} x_1 &= \mu^4 + v(\mu^3 - 2\mu^4 + 2\mu^5) \\ x_2 &= \mu^2 + 2\mu^3 - 2\mu^4 + v\mu \\ x_3 &= \mu - 2\mu^2 + 2\mu^3 + v \\ x_4 &= 4 \end{aligned}$$

The equation $F = 0$ reduces to:

$$\xi^2\,[\xi^5 - 2\xi^4 + (1 - 4\eta)\xi^3 + 6\eta\,\xi^2 + (4\eta^2 - \eta)\xi + 4\eta^2 - \eta] = 0.$$

The complete nodal curve is the cuspidal quartic:

$$x_1 = \lambda^4 \quad x_2 = \lambda^2 \quad x_3 = \lambda \quad x_4 = \quad -1.$$

It is equivalent to a nodal curve of order twenty-eight.

Let $\phi_1 = 0, \phi_2 = 0, \ldots\ldots, \phi_i = 0$ determine the same component of the nodal curve of the surface (1)(in general, $i = 1$). If in the product:—

$$\phi_1(\mu + \mu', \mu\mu') \cdot \quad \phi_2(\mu + \mu', \mu\mu') \cdot \ldots\ldots \phi_i(\mu + \mu', \mu\mu') = 0$$

we put $\mu' = \mu$, the resulting equation determines the torsal generators of the surface which have their pinch-points on the curve considered.

When a ruled surface is developable, every point on its edge of regression is a pinch-point. The corresponding component of $F = 0$ must, therefore, vanish identically for $\mu = \mu'$. That is, to the edge of regression of a developable always corresponds the conic, $\xi^2 - 4\eta = 0$.

A theory similar to that formulated here for the double curve holds also for the double developable (i. e., the developable formed by the planes containing two generators), for to the point (ξ, η) on $F(\xi, \eta) = 0$ corresponds the plane of the intersecting generators, $\frac{\xi \pm \sqrt{\xi^2 - 4\eta}}{2}$ and conversely. Thus, in the case of the quintic scroll with an oscnodal conic mentioned above, we have seen that $F = 0$ reduces to $\xi^3 = 0$. It readily follows that the double developable consists of the tangents to a quadric cone counted thrice. This surface has, in fact, the interesting property of being self-dual without belonging to a linear complex.[*]

When the surface has a rectilinear directrix, we may take the directrix to be $x_1 = x_2 = 0$, and write the equations of the surface in the form:

$$x_1 = \vartheta(\mu)a(\mu); \quad x_2 = \vartheta(\mu)b(\mu); x_3 = c(\mu) + vc'(\mu); \quad x_4 = d(\mu) + vd'(\mu),$$

wherein $a(\mu)$ and $b(\mu)$ are relatively prime.

Equation (3) now reduces to:

$$\vartheta(\mu)\,\vartheta(\mu')\begin{vmatrix} a(\mu) & a(\mu') \\ b(\mu) & b(\mu') \end{vmatrix}\begin{vmatrix} c'(\mu) & c'(\mu') \\ d'(\mu) & d'(\mu') \end{vmatrix} = 0.$$

$\vartheta(\mu) = 0$ or $\vartheta(\mu') = 0$ determines the parameters of the generators coincident with the directrix. The equation

$$\frac{1}{\mu - \mu'}\begin{vmatrix} c'(\mu) & c'(\mu') \\ d'(\mu) & d'(\mu') \end{vmatrix} = 0,$$

is satisfied by the parameters of the generators that intersect on $x_1 = x_2 = 0$. It determines the properties of the double developable other than $x_1 + \lambda x_2 = 0$. The properties of the nodal curve—other than the directrix itself—are determined by the consideration of the expression:

$$\frac{1}{\mu - \mu'}\begin{vmatrix} a(\mu) & a(\mu') \\ b(\mu) & b(\mu') \end{vmatrix} = 0.$$

The determination of the double curve and of the double developable of such a surface is, therefore, closely related to the theory of binary forms.

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[*] See Wilczynski, Math. Annalen, vol. 58, p. 249; also Bull. Amer. Math. Soc., vol. 11, p. 8. Sisam, Bull. Amer. Math. Soc., vol. 10, p. 440.

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